Lambert W function in the stability and bifurcation analysis of homographic Ricker maps

J. Leonel Rocha* and Abdel-Kaddous Taha**

*CEAUL. ADM, ISEL-Engineering Superior Institute of Lisbon, Polytechnic Institute of Lisbon, Portugal

**INSA, Federal University of Toulouse Midi-Pyrénées, Toulouse, France

Abstract. Dynamical systems of the type homographic Ricker maps are considered, which are particular cases of a new extended γ-Ricker population model: a discrete-time population model whose dynamics of the population x_n, after n generations, with $n \in \mathbf{N}$, can be defined by the difference equation $x_{n+1} = b(x_n)\ x_n^{\gamma-1}\ s(x_n)$, and written in the following form,

$$x_{n+1} = r\ \frac{x_n^{\gamma}}{\beta + x_n}\ e^{-\delta x_n} \tag{1}$$

where γ is the cooperation or Allee's effect parameter. The per-capita birth or growth function $b(x_n) = \frac{c x_n}{\beta + x_n}$ is a Holling's type II functional form or rectangular hyperbola, where $c > 0$ measures the maximal reproduction or growth rate and the ratio c/β measures the relative growth rate as the population size is smaller. The survival function for generation n or the intraspecific competition is given by $s(x_n) = e^{\mu - \delta x_n}$, where $\mu > 0$ is the density-independent death rate, $\delta > 0$ is the carrying capacity parameter, with $r = ce^{\mu}$, γ and β real parameters.

The purpose of this talk is to investigate the nonlinear dynamics properties of the homographic Ricker maps, denoted by $f(x; r, \delta, \beta)$, for some particular cases of the γ parameter. Then we study the fixed points of these homographic maps as analytical solutions of Lambert W functions. Using general properties of Lambert W functions, we establish conditions for the existence, nature and stability of the non-zero fixed points. Throughout this work, we will show how the use of Lambert W functions are useful to formalize analytical results and to represent bifurcation curves. Fold and flip bifurcation structures of the homographic Ricker maps are investigated, in which there are flip codimension-2 bifurcation points and cusp points, while some parameters evolve. Some communication areas and big bang bifurcation curves are also detected, see Fig.1. Several numerical simulations illustrate the theoretical results established.

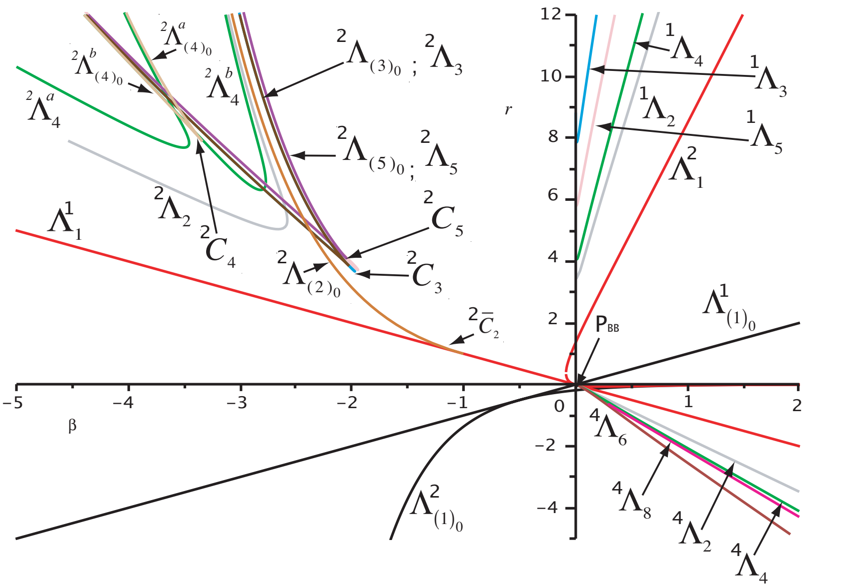

Figure 1: Bifurcation curves of the homographic Ricker map $f(x; r, \delta, \beta)$ in the $\Delta_{\beta,r}$ parameter plane at $\delta = 2$: ${}^k\Lambda^j_{(n)_0}$ are the fold bifurcation curves of the cycles of order $n = 1, 2, 3, 4, 5$, where k denotes the quadrant and j is the differentiation between curves of the same cycle; ${}^k\Lambda^j_n$ are the flip bifurcation curves of the cycles of order $n = 1, 2, 3, 4, 5, 6, 8$; ${}^2\overline{C}_2 = \Lambda^1_1 \cap {}^2\Lambda_{(2)_0}$ is a flip codimension–2 bifurcation point; ${}^2C_3$, ${}^2C_4$ and ${}^2C_5$ are the cusp points related to the cycles of order $n = 3, 4, 5$, respectively; $P_{BB} \in \tilde{\Lambda}$ is a big bang bifurcation point.

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